The Relation of Mathematics to Physics

Richard Feynman

In thinking out the applications of mathematics and physics, it is perfectly natural that the mathematics will be useful when large numbers are involved in complex situations. In biology, for example, the action of a virus on a bacterium is unmathematical. If you watch it under a microscope, a jiggling little virus finds some spot on the odd shaped bacterium- they are all different shapes - and maybe it pushes its DNA in and maybe it does not. Yet if we do the experiment with millions and millions of bacteria and viruses, then we can learn a great deal about the viruses by taking averages. We can use mathematics in the averaging, to see whether the viruses develop in the bacteria, what new strains and what percentage; and so we can study the genetics, the mutations and so forth.

To take another more trivial example, imagine an enormous board, a chequerboard to play chequers or draughts. The actual operation of any one step is not mathematical - or it is very simple in its mathematics. But you could imagine that on an enormous board, with lots and lots of pieces, some analysis of the best moves, or the good moves or bad moves, might be made by a deep kind of reasoning which would involve somebody having gone off first and thought about it in great depth. That then becomes mathematics, involving abstract reasoning. Another example is switching in computers. If you have one switch, which is either on or off, there is nothing very mathematical about that, although mathematicians like to start there with their mathematics. But with all the interconnections and wires, to figure out what a very large system will do requires mathematics.

I would like to say immediately that mathematics has a tremendous application in physics in the discussion of the detailed phenomena in complicated situations, granting the fundamental rules of the game. That is something which I would spend most of my time discussing if I were talking only about the relation of mathematics and physics. But since this is part of a series of lectures on the character of physical law I do not have time to discuss what happens in complicated situations, but will go immediately to another question, which is the character of the fundamental laws.

If we go back to our chequer game, the fundamental laws are the rules by which the chequers move. Mathematics may be applied in the complex situation to figure out what in given circumstances is a good move to make. But very little mathematics is needed for the simple fundamental character of the basic laws. They can be simply stated in English for chequers.

The strange thing about physics is that for the fundamental laws we still need mathematics. I will give two examples, one in which we really do not, and one in which we do. First, there is a law in physics called Faraday's law, which says that in electrolysis the amount of material which is deposited is proportional to the current and to the time that the current is acting. That means that the amount of material deposited is proportional to the charge which goes through the system. It sounds very mathematical, but what is actually happening is that the electrons going through the wire each carry one charge. To take a particular example, maybe to deposit one atom requires one electron to come, so the number of atoms that are deposited is necessarily equal to the number of electrons that flow, and thus proportional to the charge that goes through the wire. So that mathematically-appearing law has as its basis nothing very deep, requiring no real knowledge of mathematics. That one electron is needed for each atom in order for it to deposit itself is mathematics, I suppose, but it is not the kind of mathematics that I am talking about here.

On the other hand, take Newton's law for gravitation, which has the aspects I discussed last time. I gave you the equation:

just to impress you with the speed with which mathematical symbols can convey information. I said that the force was proportional to the product of the masses of two objects, and inversely as the square of the distance between them, and also that bodies react to forces by changing their speeds, or changing their motions, in the direction of the force by amounts proportional to the force and inversely proportional to their masses. Those are words all right, and I did not necessarily have to write the equation. Nevertheless it is kind of mathematical, and we wonder how this can be a fundamental law. What does the planet do? Does it look at the sun, see how far away it is, and decide to calculate on its internal adding machine the inverse of the square of the distance, which tells it how much to move? This is certainly no explanation of the machinery of gravitation! You might want to look further, and various people have tried to look further. Newton was originally asked about his theory - 'But it doesn't mean anything - it doesn't tell us anything'. He said, 'It tells you how it moves. That should be enough. I have told you how it moves, not why.' But people often are unsatisfied without a mechanism, and I would like to describe one theory which has been invented, among others, of the type you might want. This theory suggests that this effect is the result of large numbers of actions, which would explain why it is mathematical.

Suppose that in the world everywhere there are a lot of particles, flying through us at very high speed. They come equally in all directions - just shooting by - and once in a while they hit us in a bombardment. We, and the sun, are practically transparent for them, practically but not completely, and some of them hit. Look, then, at what would happen.

S is the sun, and E the earth. If the sun were not there, particles would be bombarding the earth from all sides, giving little impulses by the rattle, bang, bang of the few that hit. This will not shake the earth in any particular direction, because there are as many coming from one side as from the other, from top as from bottom. However, when the sun is there the particles which are coming from that direction are partly absorbed by the sun, because some of them hit the sun and do not go through. Therefore the number coming from the sun's direction towards the earth is less than the number coming from the other sides, because they meet an obstacle, the sun. It is easy to see that the farther the sun is away, of all the possible directions in which particles can come, a smaller proportion of the particles are being taken out. The sun will appear smaller - in fact inversely as the square of the distance. Therefore there will be an impulse on the earth towards the sun that varies inversely as the square of the distance. And this will be a result of large numbers of very simple operations, just hits, one after the other, from all directions. Therefore the strangeness of the mathematical relation will be very much reduced, because the fundamental operation is much simpler than calculating the inverse of the square of the distance. This design, with the particles bouncing, does the calculation. The only trouble with this scheme is that it does not work, for other reasons. Every theory that you make up has to be analysed against all possible consequences, to see if it predicts anything else. And this does predict something else. If the earth is moving, more particles will hit it from in front than from behind. (If you are running in the rain, more rain hits you in the front of the face than in the back of the head, because you are running into the rain.) So, if the earth is moving it is running into the particles coming towards it and away from the ones that are chasing it from behind. So more particles will hit it from the front than from the back, and there will be a force opposing any motion. This force would slow the earth up in its orbit, and it certainly would not have lasted the three or four billion years (at least) that it has been going around the sun. So that is the end of that theory. 'Well,' you say, 'it was a good one, and I got rid of the mathematics for a while. Maybe I could invent a better one.' Maybe you can, because nobody knows the ultimate. But up to today, from the time of Newton, no one has invented another theoretical description of the mathematical machinery behind this law which does not either say the same thing over again, or make the mathematics harder, or predict some wrong phenomena. So there is no model of the theory of gravitation today, other than the mathematical form.

If this were the only law of this character it would be interesting and rather annoying. But what turns out to be true is that the more we investigate, the more laws we find, and the deeper we penetrate nature, the more this disease persists. Every one of our laws is a purely mathematical statement in rather complex and abstruse mathematics. Newton's statement of the law of gravitation is relatively simple mathematics. It gets more and more abstruse and more and more difficult as we go on. Why? I have not the slightest idea. It is only my purpose here to tell you about this fact. The burden of the lecture is just to emphasize he fact that it is impossible to explain honestly the beauties of the laws of nature in a way that people can feel, without of their having some deep understanding of mathematics. I am sorry, but this seems to be the case. You might say, 'All right, then if there is no explanation the of law, at least tell me what the law is. Why not tell me in words instead of in symbols? Mathematics is just a language, and I want to be able to translate the language'. In fact I can, with patience, and I think I partly did. I could go a little further and explain in more detail that the equation means that if the distance is twice as far the force is one fourth as much, and so on. I could convert all the symbols into words. In other words I could be kind to the laymen as they all sit hopefully waiting for me to explain something. Different people get different reputations for their skill at explaining to the layman in layman's language these difficult and abstruse subjects. The layman then searches for book after book in the hope that he will avoid the complexities which ultimately set in, even with the best expositor of this type. He finds as he reads a generally increasing confusion, one complicated statement after another, one difficult-to-understand thing after another, all apparently disconnected from one another. It becomes obscure, and he hopes that maybe in some other book there is some explanation.... The author almost made it - maybe another fellow will make it right.

But I do not think it is possible, because mathematics is not just another language. Mathematics is a language plus reasoning; it is like a language plus logic. Mathematics is a tool for reasoning. It is in fact a big collection of the results of some person's careful thought and reasoning. By mathematics it is possible to connect one statement to another. For instance, I can say that the force is directed towards the sun. I can also tell you, as I did, that the planet moves so that if I draw a line from the sun to the planet, and draw another line at some definite period, like three weeks, later, then the area that is swung out by the planet is exactly the same as it will be in the next three weeks, and the next three weeks, and so on as it goes around the sun. I can explain both of those statements carefully, but I cannot explain why they are both the same. The apparent enormous complexities of nature, with all its funny laws and rules, each of which has been carefully explained to you, are really very closely interwoven. However, if you do not appreciate the mathematics, you cannot see, among the great variety of facts, that logic permits you to go from one to the other.

It may be unbelievable that I can demonstrate that equal areas will be swept out in equal times if the forces are directed towards the sun. So if I may, I will do one demonstration to show you that those two things really are equivalent, so that you can appreciate more than the mere statement of the two laws. I will show that the two laws are connected so that reasoning alone will bring you from one to the other, and that mathematics is just organized reasoning. Then you will appreciate the beauty of the relationship of the statements. I am going to prove the relationship that if the forces are directed towards the sun equal areas are swept out in equal times.

We start with a sun and a planet (fig. 2), and we imagine that at a certain time the planet is at position 1. It is moving in such a way that, say, one second later it has moved to position 2. If the sun did not exert a force on the planet, then, by Galileo's principle of inertia, it would keep right on going in a straight line. So after the same interval of time, the next second, it would have moved exactly the same distance in the same straight line, to the position 3. First we are going to show that if there is no force, then equal areas are swept out in equal times. I remind you that the area of a triangle is half the base times the altitude, and that the altitude is the vertical distance to the base. If the triangle is obtuse (fig. 3), then the altitude is the vertical height AD and the base is BC. Now let us compare the areas which would be swept out if the sun exerted no force whatsoever (fig- 2).

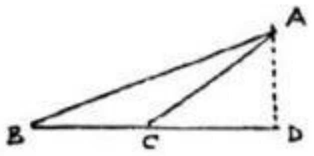

Figure 3

The two distances 1-2 and 2-3 are equal, remember. The question is, are the two areas equal ? Consider the triangle made from the sun and the two points 1 and 2. What is its area? It is the base 1-2, multiplied by half the perpendicular height from the baseline to S. What about the other triangle, the triangle in the motion from 2 to 3 ? Its area is the base 2-3, times half the perpendicular height to S. The two triangles have the same altitude, and, as I indicated, the same base, and therefore they have the same area. So far so good. If there were no force from the sun, equal areas would be swept out in equal times. But there is a force from the sun. During the interval 1-2-3 the sun is pulling and changing the motion in various directions towards itself. To get a good approximation we will take the central position, or average position, at 2, and say that the whole effect during the interval 1-3 was to change the motion by some amount in the direction of the line 2-S (fig. 4).

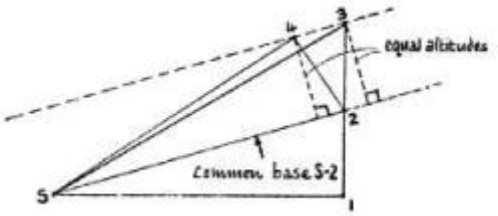

Figure 4

This means that though the particles were moving on the line 1-2, and would, were there no force, have continued to move on the same line in the next second, because of the influence of the sun the motion is altered by an amount that is poking in a direction parallel to the line 2-S. The next motion is therefore a compound of what the planet wanted to do and the change that has been induced by the action of the sun. So the planet does not really end up at position 3, but rather at position 4. Now we would like to compare the areas of the triangles 23S and 24S, and I will show you that those are equal. They have the same base, S-2. Do they have the same altitude? Sure, because they are included between parallel lines. The distance from 4 to the line S-2 is equal to the distance from 3 to line S-2 (extended). Thus the area of the triangle S24 is the same as S23. I proved earlier that S12 and S23 were equal in area, so we now know S12 = S24. So, in the actual orbital motion of the planet the areas swept out in the first second and the second second are equal. Therefore, by reasoning, we can see a connection between the fact that the force is towards the sun, and the fact that the areas are equal. Isn't that ingenious ? I borrowed it straight from Newton. It comes right out of the Principia, diagram and all. Only the letters are different, because he wrote in Latin and these are Arabic numerals.

Newton made all the proofs in his book geometrical. Today we do not use that kind of reasoning. We use a kind of analytic reasoning with symbols. It requires ingenuity to draw the correct triangles, to notice about the areas, and to figure out how to do this. But there have been improvements in the methods of analysis, which are faster and more efficient. I want to show what this looks like in the notation of the more modern mathematics, where you do nothing but write a lot of symbols to figure it out.

We want to talk about how fast the area changes, and we represent that by $\dot{A}$. The area changes when the radius is swinging, and it is the component of velocity at right angles to the radius, times the radius, that tells us how fast the area changes. So this is the component of the radial distance multiplied by the velocity, or rate of change of the distance.

The question now is whether the rate of change of area itself changes. The principle is that the rate of change of the area is not supposed to change. So we differentiate this again, and this means some little trick about putting dots in the right place, that is all. You have to learn the tricks; it is just a series of rules that people have found out that are very useful for such a thing. We write:

This first term says to take the component of the velocity at right angles to the velocity. It is zero; the velocity is in the same direction as itself. The acceleration, which is the second derivative, r with two dots, or the derivative of the velocity, is the force divided by the mass.

This says therefore that the rate of change of the rate of change of the area is the component of force at right angles to the radius, but if the force is in the direction of the radius,as Newton said, then there is no force at right angles to the radius, and that means that the rate of change of area does not change. This merely illustrates the power of analysis with different kinds of notation. Newton knew how to do this, more or less, with slightly different notations; but he wrote everything in the geometrical form, because he tried to make it possible for people to read his papers. He invented the calculus, which is the kind of mathematics Ihave just shown.

This is a good illustration of the relation of mathematics to physics. When the problems in physics become difficult we may often look to the mathematicians, who may already have studied such things and have prepared a line of reasoning for us to follow. On the other hand they may not have, in which case we have to invent our own line of reasoning, which we then pass back to the mathematicians. Everybody who reasons carefully about anything is making a contribution to the knowledge of what happens when you think about something, and if you abstract it away and send it to the Department of Mathematics they put it in books as a branch of mathematics. Mathematics, then, is a way of going from one set of statements to another. It is evidently useful in physics, because we have these different ways in which we can speak of things, and mathematics permits us to develop consequences, to analyse the situations, and to change the laws in different ways to connect the various statements. In fact the total amount that a physicist knows is very little. He has only to remember the rules to get him from one place to another and he is all right, because all the various statements about equal times, the force being in the direction of the radius, and so on, are all interconnected by reasoning.

Now an interesting question comes up. Is there a place to begin to deduce the whole works ? Is there some particular pattern or order in nature by which we can understand that one set of statements is more fundamental and one set of statements more consequential ? There are two kinds of ways of looking at mathematics, which for the purpose of this lecture I will call the Babylonian tradition and the Greek tradition. In Babylonian schools in mathematics the student would learn something by doing a large number of examples until he caught on to the general rule. Also he would know a large amount of geometry, a lot of the properties of circles, the theorem of Pythagoras, formulae for the areas of cubes and triangles; in addition, some degree of argument was available to go from one thing to another. Tables of numerical quantities were available so that they could solve elaborate equations. Everything was prepared for calculating things out. But Euclid discovered that there was a way in which all of the theorems of geometry could be ordered from a set of axioms that were particularly simple. The Babylonian attitude - or what I call Babylonian mathematics - is that you know all of the various theorems and many of the connections in between, but you have never fully realized that it could all come up from a bunch of axioms. The most modern mathematics concentrates on axioms and demonstrations within a very definite framework of conventions of what is acceptable and what is not acceptable as axioms. Modern geometry takes something like Euclid's axioms, modified to be more perfect, and then shows the deduction of the system. For instance, it would not be expected that a theorem like Pythagoras's (that the sum of the areas of squares put on two sides of a right- angled triangle is equal to the area of the square on the hypotenuse) should be an axiom. On the other hand, from another point of view of geometry, that of Descartes, the Pythagorean theorem is an axiom.

So the first thing we have to accept is that even in mathematics you can start in different places. If all these various theorems are interconnected by reasoning there is no real way to say 'These are the most fundamental axioms', because if you were told something different instead you could also run the reasoning the other way. It is like a bridge with lots of members, and it is over-connected; if pieces have dropped out you can reconnect it another way. The mathematical tradition of today is to start with some particular ideas which are chosen by some kind of convention to be axioms, and then to build up the structure from there. What I have called the Babylonian idea is to say, 'I happen to know this, and I happen to know that, and maybe I know that; and I work everything out from there. Tomorrow I may forget that this is true, but remember that something else is true, so I can reconstruct it all again. I am never quite sure of where I am supposed to begin or where I am supposed to end. I just remember enough all the time so that as the memory fades and some of the pieces fall out I can put the thing back together again every day'. The method of always starting from the axioms is not very efficient in obtaining theorems. In working something out in geometry you are not very efficient if each time you have to start back at the axioms. If you have to remember a few things in geometry you can always get somewhere else, but it is much more efficient to do it the other way. To decide which are the best axioms is not necessarily the most efficient way of getting around in the territory. In physics we need the Babylonian method, and not the Euclidian or Greek method. I would like to explain why.

The problem in the Euclidian method is to make something about the axioms a little more interesting or important. But in the case of gravitation, for example, the question we are asking is: is it more important, more basic, or is it a better axiom, to say that the force is towards the sun, or to say that equal areas are swept out in equal times ? From one point of view the force statement is better. If I state what the forces are I can deal with a system with many particles in which the orbits are no longer ellipses, because the force statement tells me about the pull of one on the other. In this case the theorem about equal areas fails. Therefore I think that the force law ought to be an axiom instead of the other. On the other hand, the principle of equal areas can be generalized, in a system of a large number of particles, to another theorem. It is rather complicated to say, and not quite as pretty as the original statement about equal areas, but it is obviously its offspring. Take a system with a large number of particles, perhaps Jupiter, Saturn, the Sun, and lots of stars, all interacting with each other, and look at it from far away projected on a plane (fig. 5). The particles are all moving in various directions, and we take any point and calculate how much area is being swept out by the radius from this point to each of the particles. In this calculation the masses which are heavier count more strongly; if one particle is twice as heavy as another its area will count twice as much. So we count each of the areas swept out in proportion to the mass that is doing the sweeping, add them all together, and the resulting total is not changing in time. That total is called the angular momentum, and this is called the law of conservation of angular momentum. Conservation just means that it does not change.

One of the consequences of this is as follows. Imagine a lot of stars falling together to form a nebula, or galaxy. At first they are very far out, on long radii from the centre, moving slowly and allowing a small amount of area to be generated. As they come closer the distances to the centre will shorten, and when they are very far in the radii will be very small, so in order to produce the same area per second they will have to move a great deal faster. You will see then that as the stars come in they will swing and swirl around faster and faster, and thus we can roughly understand the qualitative shape of the spiral nebulae. In the same way we can understand how a skater spins. He starts with his leg out, moving slowly, and as he pulls his leg in he spins faster. When the leg is out it is contributing a certain amount of area per second, and then when he brings his leg in he has to spin much faster to produce the same amount of area. But I did not prove it for the skater: the skater uses muscle force, and gravity is a different force. Yet it is true for the skater.

Now we have a problem. We can deduce often from one part of physics, like the Law of Gravitation, a principle which turns out to be much more valid than the derivation. This does not happen in mathematics; theorems do not come out in places where they are not supposed to be. In other words, if we were to say that the postulate of physics was the equal area law of gravitation, then we could deduce the conservation of angular momentum, but only for gravitation. Yet we discover experimentally that the conservation of angular momentum is a much wider thing. Newton had other postulates by which he could get the more general conservation law of angular momentum. But these Newtonian laws were wrong. There are no forces, it is all a lot of boloney, the particles do not have orbits, and so on. Yet the analogue, the exact transformation of this principle about the areas and the conservation of angular momentum, is true. It works for atomic motions in quantum mechanics, and, as far as we can tell, it is still exact today. We have these wide principles which sweep across the different laws, and if we take the derivation too seriously, and feel that one is only valid because another is valid, then we cannot understand the interconnections of the different branches of physics. Someday, when physics is complete and we know all the laws, we may be able to start with some axioms, and no doubt somebody will figure out a particular way of doing it so that everything else can be deduced. But while we do not know all the laws, we can use some to make guesses at theorems which extend beyond the proof. In order to understand physics one must always have a neat balance, and contain in one's head all of the various propositions and their interrelationships, because the laws often extend beyond the range of their deductions. This will only have no importance when all the laws are known.

Another thing, a very strange one, that is interesting in the relation of mathematics to physics is the fact that by mathematical arguments you can show that it is possible to start from many apparently different starting points, and yet come to the same thing. That is pretty clear. If you have axioms, you can instead use some of the theorems; but actually the physical laws are so delicately constructed that the different but equivalent statements of them have such qualitatively different characters, and this makes them very interesting. To illustrate this I am going to state the law of gravitation in three different ways, all of which are exactly equivalent but sound completely different. The first statement is that there are forces between objects, according to the equation which I have given you before.

Each object, when it sees the force on it, accelerates or changes its motion, at a certain amount per second. It is the regular way of stating the law, I call it Newton's law. This statement of the law says that the force depends on something at a finite distance away. It has what we call an unlocal quality. The force on one object depends on where another one is some distance away.

You may not like the idea of action at a distance. How can this object know what is going on over there? So there is another way of stating the laws, which is very strange, called the field way. It is hard to explain, but I want to give you some rough idea of what it is like. It says a completely different thing. There is a number at every point in space (I know it is a number, not a mechanism: that is the trouble with physics, it must be mathematical), and the numbers change when you go from place to place. If an object is placed at a point in space, the force on it is in the direction in which that number changes most rapidly (I will give it its usual name, the potential, the force is in the direction in which the potential changes). Further, the force is proportional to how fast the potential changes as you move. That is one part of the statement, but it is not enough, because I have yet to tell you how to determine the way in which the potential varies. I could say the potential varies inversely as the distance from each object, but that is back to the reaction-at-a-distance idea. You can state the law in another way, which says that you do not have to know what is going on anywhere outside a little ball. If you want to know what the potential is at the centre of the ball, you need only tell me what it is on the surface of the ball, however small. You do not have to look outside, you just tell me what it is in the neighbourhood, and how much mass there is in the ball. The rule is this. The potential at the centre is equal to the average of the potential on the surface of the ball, minus the same constant, G, as we had in the other equation, divided by twice the radius of the ball (which we will call a), and then multiplied by the mass inside the ball, if the ball is small enough.
You see that this law is different from the other, because it tells what happens at one point in terms of what happens very close by. Newton's law tells what happens at one time in terms of what happens at another instant. It gives from instant to instant how to work it out, but in space leaps from place to place. The second statement is both local in time and local in space, because it depends only on what is in the neighbourhood. But both statements are exactly equivalent mathematically.

There is another completely different way of stating this, different in the philosophy and the qualitative ideas involved. If you do not like action at a distance I have shown you can get away without it. Now I want to show you a statement which is philosophically the exact opposite. In this there is no discussion at all about how the thing works its way from place to place; the whole is contained in an overall statement, as follows. When you have a number of particles, and you want to know how one moves from one place to another, you do it by inventing a possible motion that gets from one place to the other in a given amount of time (fig. 6). Say the particle wants to go from X to Y in an hour, and you want to know by what route it can go. What you do is to invent various curves, and calculate on each curve a certain quantity. (I do not want to tell you what the quantity is, but for those who have heard of these terms the quantity on each route is the average of the difference between the kinetic and the potential energy.) If you calculate this quantity for one route, and then for another, you will get a different number for each route. There is one route which gives the least possible number, however, and that is the route that the particle in nature actually takes! We are now describing the actual motion, the ellipse, by saying something about the whole curve. We have lost the idea of causality, that the particle feels the pull and moves in accordance with it. Instead of that, in some grand fashion it smells all the curves, all the possibilities, and decides which one to take (by choosing that for which our quantity is least).

This is an example of the wide range of beautiful ways of describing nature. When people say that nature must have causality, you can use Newton's law; or if they say that nature must be stated in terms of a minimum principle, you talk about it this last way; or if they insist that nature must have a local field - sure, you can do that. The question is: which one is right? If these various alternatives are not exactly equivalent mathematically, if for certain ones there will be different consequences than for others, then all we have to do is to experiment to find out which way nature actually chooses to do it. People may come along and argue philosophically that they like one better than another; but we have learned from much experience that all philosophical intuitions about what nature is going to do fail. One just has to work out all the possibilities, and try all the alternatives. But in the particular case I am talking about the theories are exactly equivalent. Mathematically each of the three different formulations, Newton's law, the local field method and the minimum principle, gives exactly the same consequences. What do we do then? You will read in all the books that we cannot decide scientifically on one or the other. That is true. They are equivalent scientifically. It is impossible to make a decision, because there is no experimental way to distinguish between them if all the consequences are the same. But psychologically they are very different in two ways. First, philosophically you like them or do not like them; and training is the only way to beat that disease. Second, psychologically they are different because they are completely unequivalent when you are trying to guess new laws.

As long as physics is incomplete, and we are trying to understand the other laws, then the different possible formulations may give clues about what might happen in other circumstances. In that case they are no longer equivalent, psychologically, in suggesting to us guesses about what the laws may look like in a wider situation. To give an example, Einstein realized that electrical signals could not propagate faster than the speed of light. He guessed that it was a general principle. (This is the same guessing game as taking the angular momentum and extending it from one case where you have proved it, to the rest of the phenomena of the universe.) He guessed that it was true of everything, and he guessed that it would be true of gravitation. If signals cannot go any faster than the speed of light, then it turns out that the method of describing the forces instantaneously is very poor. So in Einstein's generalization of gravitation Newton's method of describing physics is hopelessly inadequate and enormously complicated, whereas the field method is neat and simple, and so is the minimum principle. We have not decided between the last two yet.

In fact it turns out that in quantum mechanics neither is right in exactly the way I have stated them, but the fact that a minimum principle exists turns out to be a consequence of the fact that on a small scale particles obey quantum mechanics. The best law, as at present understood, is really a combination of the two in which we use minimum principles plus local laws. At present we believe that the laws of physics have to have the local character and also the minimum principle, but we do not really know. If you have a structure that is only partly accurate, and something is going to fail, then if you write it with just the right axioms maybe only one axiom fails and the rest remain, you need only change one little thing. But if you write it with another set of axioms they may all collapse, because they all lean on that one thing that fails. We cannot tell ahead of time, without some intuition, which is the best way to write it so that we can find out the new situation. We must always keep ll the alternative ways of looking at a thing in our heads; so physicists do Babylonian mathematics, and pay but little attention to the precise reasoning from fixed axioms.

One of the amazing characteristics of nature is the variety of interpretational schemes which is possible. It turns out that it is only possible because the laws are just so, special and delicate. For instance, that the law is the inverse square is what permits it to become local; if it were the inverse cube it could not be done that way. At the other end of the equation, the fact that the force is related to the rate of change of velocity is what permits the minimum principle way of writing the laws. If, for instance, the force were proportional to the rate of change of position instead of velocity, then you could not write it in that way. If you modify the laws much you find that you can only write them in fewer ways. I always find that mysterious, and I do not understand the reason why it is that the correct laws of physics seem to be expressible in such a tremendous variety of ways. They seem to be able to get through several wickets at the same time.

I should like to say a few things on the relation of mathematics and physics which are a little more general. Mathematicians are only dealing with the structure of reasoning, and they do not really care what they are talking about. They do not even need to know what they are talking about, or, as they themselves say, whether what they say is true. I will explain that. You state the axioms, such-and-such is so, and such-and-such is so. What then? The logic can be carried out without knowing what the such-and-such words mean. If the statements about the axioms are carefully formulated and complete enough, it is not necessary for the man who is doing the reasoning to have any knowledge of the meaning of the words in order to deduce new conclusions in the same language. If I use the word triangle in one of the axioms there will be a statement about triangles in the conclusion, whereas the man who is doing the reasoning may not know what a triangle is. But I can read his reasoning back and say, Triangle, that is just a three-sided what- have-you, which is so-and-so', and then I know his new facts. In other words, mathematicians prepare abstract reasoning ready to be used if you have a set of axioms about the real world. But the physicist has meaning to all his phrases. That is a very important thing that a lot of people who come to physics by way of mathematics do not appreciate. Physics is not mathematics, and mathematics is not physics. One helps the other. But in physics you have to have an understanding of the connection of words with the real world. It is necessary at the end to translate what you have figured out into English, into the world, into the blocks of copper and glass that you are going to do the experiments with. Only in that way can you find out whether the consequences are true. This is a problem which is not a problem of mathematics at all.

Of course it is obvious that the mathematical reasonings which have been developed are of great power and use for physicists. On the other hand, sometimes the physicists' reasoning is useful for mathematicians.

Mathematicians like to make their reasoning as general as possible. If I say to them, 'I want to talk about ordinary three dimensional space', they say 'If you have a space of n dimensions, then here are the theorems'. 'But I only want the case 3', 'Well, substitute n = 3.'! So it turns out that many of the complicated theorems they have are much simpler when adapted to a special case. The physicist is always interested in the special case; he is never interested in the general case. He is talking about something; he is not talking abstractly about anything. He wants to discuss the gravity law in three dimensions; he never wants the arbitrary force case in n dimensions. So a certain amount of reducing is necessary, because the mathematicians have prepared these things for a wide range of problems. This is very useful, and later on it always turns out that the poor physicist has to come back and say, 'Excuse me, when you wanted to tell me about four dimensions ...'

When you know what it is you are talking about, that some symbols represent forces, others masses, inertia, and so on, then you can use a lot of commonsense, seat-of-the-pants feeling about the world. You have seen various things, and you know more or less how the phenomenon is going to behave. But the poor mathematician translates it into equations, and as the symbols do not mean anything to him he has no guide but precise mathematical rigour and care in the argument. The physicist, who knows more or less how the answer is going to come out, can sort of guess part way, and so go along rather rapidly. The mathematical rigour of great precision is not very useful in physics. But one should not criticize the mathematicians on this score. It is not necessary that just because something would be useful to physics they have to do it that way. They are doing their own job. If you want something else, then you work it out for yourself.

The next question is whether, when trying to guess a new law, we should use the seat-of-the-pants feeling and philosophical principles – 'I don't like the minimum principle', or'I do like the minimum principle', 'I don't like action at a distance', or 'I do like action at a distance'. To what extent do models help? It is interesting that very often models do help, and most physics teachers try to teach how to use models and to get a good physical feel for how things are going to work out. But it always turns out that the greatest discoveries abstract away from the model and the model never does any good. Maxwell's discovery of electrodynamics was first made with a lot of imaginary wheels and idlers in space. But when you get rid of all the idlers and things in space the thing is O.K. Dirac discovered the correct laws for relativity quantum mechanics simply by guessing the equation. The method of guessing the equation seems to be a pretty effective way of guessing new laws. This shows again that mathematics is a deep way of expressing nature, and any attempt to express nature in philosophical principles, or in seat-of-the-pants mechanical feelings, is not an efficient way.

It always bothers me that, according to the laws as we understand them today, it takes a computing machine an infinite number of logical operations to figure out what goes on in no matter how tiny a region of space, and no matter how tiny a region of time. How can all that be going on in that tiny space? Why should it take an infinite amount of logic to figure out what one tiny piece of space/time is going to do? So I have often made the hypothesis that ultimately physics will not require a mathematical statement, that in the end the machinery will be revealed, and the laws will turn out to be simple, like the chequer board with all its apparent complexities. But this speculation is of the same nature as those other people make - 'I like it', 'I don't like it', - and it is not good to be too prejudiced about these things.

To summarize, I would use the words of Jeans, who said that 'the Great Architect seems to be a mathematician'. To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature. C. P. Snow talked about two cultures. I really think that those two cultures separate people who have and people who have not had this experience of understanding mathematics well enough to appreciate nature once.

It is too bad that it has to be mathematics, and that mathematics is hard for some people. It is reputed

-1 do not know if it is true - that when one of the kings was trying to learn geometry from Euclid he complained that it was difficult. And Euclid said, There is no royal road to geometry'. And there is no royal road. Physicists cannot make a conversion to any other language. If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in. She offers her information only in one form; we are not so unhumble as to demand that she change before we pay any attention.

All the intellectual arguments that you can make will not communicate to deaf ears what the experience of music really is. In the same way all the intellectual arguments in the world will not convey an understanding of nature to those of 'the other culture'. Philosophers may try to teach you by telling you qualitatively about nature. I am trying to describe her. But it is not getting across because it is impossible. Perhaps it is because their horizons are limited in this way that some people are able to imagine that the centre of the universe is man.